Eks. X er $\begin{cases} \text{binomisk} \\ \text{poisson} \\ \text{normal} \\ \text{eksponential} \end{cases}$ Kva med $\sum X_i$

Def. 7.2

Momentgenererande funksjon er definert ved

$$M_X(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} f(x), & X \text{ diskret} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, & X \text{ kontinuerleg} \end{cases}$$

Vi har $$\frac{d^n M_X(t)}{dt^n} = \begin{cases} \sum_x x^n e^{tx} f(x), & X \text{ diskret} \\ \int_{-\infty}^{\infty} x^n e^{tx} f(x)\,dx, & X \text{ kontinuerleg.} \end{cases}$$

slik at $$E[X^n] = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0} = \mu_n'$$

Eksempel

$X \sim$ Bernoulli fordelt d:

x	0	1
$P(X=x)$	$1-p$	p

$$M_X(t) = E[e^{tX}] = e^{t\cdot 0}(1-p) + e^t p = 1-p+pe^t$$

$$\frac{d\,M_X(t)}{dt} = pe^t. \Rightarrow E[X] = \left.\frac{d\,M_X(t)}{dt}\right|_{t=0} = p.$$

$$E[X^2] = \left.\frac{d^2 M_X(t)}{dt^2}\right|_{t=0} = \left. pe^t \right|_{t=0} = p \Rightarrow Var[X] = p - p^2 = p(1-p)$$

$X \sim$ geometrisk fordelt med sannsyn p d: $P(X=x) = (1-p)^{x-1} \cdot p$, $x=1,2,3$

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx}(1-p)^{x-1} p = pe^t \sum_{x=1}^{\infty} e^{t(x-1)}(1-p)^{x-1}$$

$$= pe^t \sum_{x=1}^{\infty} (e^t(1-p))^{x-1} = \frac{pe^t}{1-e^t(1-p)}, \quad e^t(1-p) < 1 \Leftrightarrow t < -\ln(1-p)$$

$$M_X'(t) = \frac{pe^t}{(1-e^t(1-p))^2} \Rightarrow M_X'(0) = \frac{1}{p} = E[X]$$

$$M_X''(t) = \frac{pe^t(1+e^t(1-p))}{(1-e^t(1-p))^3} \Rightarrow M_X''(0) = \frac{2-p}{p^2}$$

$$\Rightarrow Var[X] = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

$X \sim$ eksponentialfordelt med parameter $\frac{1}{\beta}$

ɔ: $f_X(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}}, & x > 0 \\ 0, & \text{ellers} \end{cases}$

$$M_X(t) = \int_0^{\infty} e^{tx} \frac{1}{\beta} e^{-\frac{x}{\beta}} dx = \frac{1}{\beta} \int_0^{\infty} e^{(t-\frac{1}{\beta})x} dx = \frac{1}{\beta} \left[\frac{1}{t-\frac{1}{\beta}} e^{(t-\frac{1}{\beta})x} \right]_0^{\infty}$$

$$= \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}$$

$X \sim N(\mu, \sigma^2)$

$$\Rightarrow M_X(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Vi har: $tx - \frac{(x-\mu)^2}{2\sigma^2} = \frac{-(x^2 - 2x(\mu+\sigma^2 t) + \mu^2)}{2\sigma^2}$

$$= \frac{-((x-(\mu+\sigma^2 t))^2 - \mu^2 - 2\mu\sigma^2 t - \sigma^4 t^2 + \mu^2)}{2\sigma^2}$$

$$= -\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2} + \mu t + \frac{\sigma^2 t^2}{2}.$$

Dette gjev $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}}\, dx$

$z = \frac{x-(\mu+\sigma t)}{\sigma}$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz}_{1} = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Eigenskapar til momentgenererande funksjonar.

7.8. $M_{X+a}(t) = E[e^{t(X+a)}] = e^{at} M_X(t)$

7.9. $M_{aX}(t) = E[e^{taX}] = M_X(at)$

7.10. $X_1, X_2, \ldots, X_n$ uavh. La $Y = \sum_{i=1}^{n} X_i$

$$M_Y(t) = E\left[e^{t\sum_{i=1}^{n} X_i}\right] = \int_{x_n} \cdots \int_{x_1} e^{t\sum x_i} f(x_1, x_2, \ldots, x_n)\, dx_1 \cdots dx_n$$

$$= \int e^{tx_1} f_{X_1}(x_1)\, dx_1 \cdots \int e^{tx_n} f_{X_n}(x_n)\, dx_n = M_{X_1}(t) \cdots M_{X_n}(t)$$

Teorem 7.7

Dersom to stokastiske variable X og Y har same momentgenererande funksjon, har dei og same fordeling.

Eks. $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ og uavh.

La $Y = a_1 X_1 + a_2 X_2$

$$M_Y(t) = M_{X_1}(a_1 t) \cdot M_{X_2}(a_2 t) = e^{\mu_1 a_1 t + \frac{\sigma_1^2 a_1^2 t^2}{2}} \cdot e^{\mu_2 a_2 t + \frac{\sigma_2^2 a_2^2 t^2}{2}}$$

$$= e^{(a_1\mu_1 + a_2\mu_2)t + (a_1^2\sigma_1^2 + a_2^2\sigma_2^2)\frac{t^2}{2}}$$

o: $Y \sim N(a_1\mu_1 + a_2\mu_2,\ a_1^2\sigma_1^2 + a_2^2\sigma_2^2)$

Generelt 7.11: $X_i \sim N(\mu_i, \sigma_i^2)$ og uavh $\Rightarrow \sum_{i=1}^{n} a_i^2 X_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i,\ \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

Resultat

La $X \sim B(n,p)$, $X = \sum_{j=1}^{n} I_j$ der I_j er Bernoulli fordelt.

$\Rightarrow M_X(t) = (1-p+pe^t)^n$

La $X_1 \sim B(n_1, p)$ og $X_2 \sim B(n_2, p)$ og uavh.

Då er $Y = X_1 + X_2 \sim B(n_1+n_2, p)$

Bevis $M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) = (1-p+pe^t)^{n_1+n_2}$

Resultat.

$X_1 \sim P(\lambda_1 t_1)$, $X_2 \sim P(\lambda_2 t_2)$ og uavh.

Då er $Y = X_1 + X_2 \sim P(\lambda_1 t_1 + \lambda_2 t_2)$

Bevis: La $X \sim P(\lambda t_1)$

$$\Rightarrow M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{(\lambda t_1)^x e^{-\lambda t_1}}{x!} = e^{-\lambda t_1} \sum_{x=0}^{\infty} \frac{(e^t \lambda t_1)^x}{x!} = e^{-\lambda t_1} e^{\lambda t_1 e^t} = \underline{e^{\lambda t_1 (e^t - 1)}}$$

$$M_Y(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1 t_1 (e^t - 1)} \cdot e^{\lambda_2 t_2 (e^t - 1)} = \underline{e^{(\lambda_1 t_1 + \lambda_2 t_2)(e^t - 1)}}$$

Ordningsvariable

La oss sjå på eit seriesystem samansett av komponentar med uavhengige og identisk fordelte levetider.

$X_1, \dots, X_n$. $\quad P(X_i \le x) = F_X(x)$, $i = 1, 2, \dots, n$

$-\boxed{X_1}-\boxed{X_2}- \cdots -\boxed{X_n}$ $\qquad$ Eks. julelys.

La U vere levetida til systemet. $U = \min(X_1, X_2, \dots, X_n)$

$$P(U \le u) = 1 - P(U > u) = 1 - P(X_1 > u, X_2 > u, \dots, X_n > u)$$

$$= 1 - P(X_1 > u) \cdot P(X_2 > u) \cdots P(X_n > u) = 1 - (1 - F_X(u))^n$$

$$f_U(u) = n(1 - F_X(u))^{n-1} f_X(u)$$

Eks.

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \quad i = 1, 2, \dots, n \\ 0, & \text{elles} \end{cases}$$

ɔ: eksponentialfordelt med parameter $\beta = \frac{1}{\lambda}$

La $U = \min(X_1, X_2, \dots, X_n)$

$$\Rightarrow F_U(u) = \begin{cases} 1 - (1 - 1 + e^{-\lambda u})^n, & u > 0 \\ 0, & u \le 0 \end{cases} = \begin{cases} 1 - e^{-n\lambda x}, & u > 0 \\ 0, & u \le 0 \end{cases}$$

ɔ: eksponentialfordelt med parameter $\beta = \frac{1}{n\lambda}$

og $E[U] = \frac{1}{n\lambda} = \frac{E[X_i]}{n}$